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Hadamard matrices of generalized quaternion type

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Abstract

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Let G be a semi-direct product of a cyclic group of an odd order by a generalized quaternion group Q_s. We consider the ring $\mathcal{R}$ obtained from the group ring $\mathbb{Z}G$ by identifying the elements ± 1 in the center of Q_s with ± 1 of the rational integer ring $\mathbb{Z}$. If the right regular representation of matrix of an element in $\mathcal{R}$ is an Hadamard matrix, we call this an Hadamard matrix of generalized quaternion type.

An Hadamard matrix generated by the Paley type 1 matrix is Seidel-equivalent to an Hadamard matrix of generalized quaternion type, bound by some conditions. When the order of generalized quaternion group is minimum, i.e. when Q_s is the quaternion group, then an Hadamard matrix of generalized quaternion type is exactly an Hadamard matrix of type Q. See Ito [2]. Moreover if the four component matrices of an Hadamard matrix of type Q are symmetric, then this becomes an Hadamard matrix of Williamson type.

The purpose of this paper is to prove the existence of some infinite series of Hadamard matrices of generalized quaternion type. The theory of the relative Gauss sum is very important for the construction of our infinite series.

In the last section, we give examples of Hadamard matrices of generalized quaternion type of order 24 in detail.

1. Definitions

A generalized quaternion group Q_s of order 2^{s+2} is a group generated by the two elements ρ, j such that

$$\rho^{2^{s+1}} = 1, \qquad j^2 = \rho^{2^s}, \qquad j\rho j^{-1} = \rho^{-1}.$$

Let G be a semi-direct product of a cyclic group of an odd order n by the generalized quaternion group Q_s of order 2^{s+2}. That is, G is generated by ρ, ζ and j with the relations

$$\rho^{2^s} = -1, \qquad j^2 = -1, \qquad j\rho j^{-1} = \rho^{-1}, \qquad \rho\zeta\rho^{-1} = \zeta, \qquad j\zeta j^{-1} = \zeta^{-1}, \qquad \zeta^n = 1.$$

We consider the ring $\mathcal{R}$ obtained from the group ring $\mathbb{Z}G$ by identifying the elements ± 1 in the center of Q_s with ± 1 of the rational integer ring $\mathbb{Z}$. Put $\mathcal{H} = \{\rho^k \zeta^l : 0 \leqq k \leqq 2^s - 1,\ 0 \leqq l \leqq n-1\}$ and choose the basis $\mathcal{L} = \mathcal{H} \cup \mathcal{H}j$ of $\mathcal{R}$. An element ξ in $\mathcal{R}$ takes the following form

$$\xi = \sum_{k=0}^{2N-1} \sum_{l=0}^{n-1} a_{k,l} \zeta^l \rho^k + \sum_{k=0}^{2N-1} \sum_{l=0}^{n-1} b_{k,l} \zeta^l \rho^k j = \alpha + \beta j, \quad N = 2^{s-1}, \tag{1}$$

where

$$\alpha = \sum_{k=0}^{2N-1} \sum_{l=0}^{n-1} \alpha_{k,l} \zeta^l \rho^k \quad \text{and} \quad \beta = \sum_{k=0}^{2N-1} \sum_{l=0}^{n-1} b_{k,l} \zeta^l \rho^k.$$

We define the conjugate $\bar{\xi} = \bar{\alpha} - \beta j$ of $\xi = \alpha + \beta j$ based on the automorphism $\tau : \rho \to \rho^{-1},\ \zeta \to \zeta^{-1}$ of G. Furthermore we define the norm $\mathcal{N}(\xi) = \xi\bar{\xi}$, so that

$$\mathcal{N}(\xi) = \alpha\bar{\alpha} + \beta\bar{\beta},$$
$$\mathcal{N}(\xi\eta) = \mathcal{N}(\xi)\mathcal{N}(\eta) \quad \text{for } \xi, \eta \in \mathcal{R}.$$

For an arbitrary element $\xi \in \mathcal{R}$, we construct the right regular representation matrix $R(\xi)$, defined by

$$(\rho^k \zeta^l \xi) = R(\xi)(\rho^k \zeta^l).$$

More precisely, for an element ξ of $\mathcal{R}$ with the form (1) the right regular representation matrix $R(\xi)$ is given by

$$R(\xi) = \begin{pmatrix} \mathcal{A} & \mathcal{B} \\ -\mathcal{B}^* & \mathcal{A}^* \end{pmatrix},$$

$$\mathcal{A} = \begin{pmatrix} A_0 & A_1 & \cdots & A_{2N-1} \\ -A_{2N-1} & A_0 & \cdots & A_{2N-2} \\ -A_{2N-2} & -A_{2N-1} & \cdots & A_{2N-3} \\ \vdots & \vdots & & \vdots \\ -A_1 & -A_2 & \cdots & A_0 \end{pmatrix},$$

$$\mathcal{B} = \begin{pmatrix} B_0 & B_1 & \cdots & B_{2N-1} \\ -B_{2N-1} & B_0 & \cdots & B_{2N-2} \\ -B_{2N-2} & -B_{2N-1} & \cdots & B_{2N-3} \\ \vdots & \vdots & & \vdots \\ -B_1 & -B_2 & \cdots & B_0 \end{pmatrix}.$$

where A^* and B^* are the transpose of A and B, and $A_k = \sum_{l=0}^{n-1} a_{k,l} T^l$ and $B_k = \sum_{l=0}^{n-1} b_{k,l} T^l$ are the circulant matrices of order n where T denotes the basic

circulant matrix of order n

$$T = \begin{pmatrix} 0 & 1 & 0 & & & 0 \\ 0 & 0 & 1 & & & 0 \\ & & & 1 & & \\ & & & & 1 & \\ & & & & & \ddots \\ & & & & & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Since $R(\bar{\xi}) = R(\xi)^*$, we have

$$R(\xi)R(\xi)^* = R(\xi)R(\bar{\xi}) = R(\xi\bar{\xi}) = \begin{pmatrix} \mathscr{A}\mathscr{A}^* + \mathscr{B}\mathscr{B}^* & 0 \\ 0 & \mathscr{A}\mathscr{A}^* + \mathscr{B}\mathscr{B}^* \end{pmatrix}.$$

Definition. If an element in $\mathscr{R}$ which is given by the equation (1) above satisfies

(i) all the coefficients $a_{k,l}$, $b_{k,l}$ are from $\{1, -1\}$, and

(ii) $\mathscr{N}(\xi) = 2^{s+1}n = 4nN$,

then the right regular representation matrix $R(\xi)$ becomes an Hadamard matrix of order $2^{s+1}n = 4nN$, which is called an *Hadamard matrix of generalized quaternion type.*

Similarly if the following conditions are satisfied:

(iii) $a_{k,k} = 0$, and all other coefficients $a_{k,l}$, $b_{k,l}$ are from $\{1, -1\}$, and

(iv) $\mathscr{N}(\xi) = 2^{s+1}n - 1 = 4nN - 1$,

then $R(\xi)$ is a C-matrix of order $2^{s+1}n = 4nN$, which we call a *C-matrix of generalized quaternion type.*

For the definition of C-matrices see [5].

We abbreviate *generalized quaternion type* as GQ *type* for convenience sake.

Let us express the conditions (i), (ii) in terms of the component matrices A_k and B_k:

$$\sum_{k=0}^{2N-1} A_k A_k^* + \sum_{k=0}^{2N-1} B_k B_k^* = 4nNI,$$

$$\sum_{k=0}^{t-1} (A_k A_{2N-t+k}^* + B_k B_{2N-t+k}^*) - \sum_{k=0}^{2N-t-1} (A_k^* A_{k+t} + B_k^* B_{k+t}) = 0 \quad \text{for } 1 \leqslant t \leqslant 2N-1.$$

In particular in case $N = 1$, the conditions will become

$$A_0A_0^* + A_1A_1^* + B_0B_0^* + B_1B_1^* = 4nI,$$

$$A_0A_1^* - A_1A_0^* + B_0B_1^* - B_1B_0^* = 0.$$

It is plain that in this case $R(\xi)$ gives an Hadamard matrix of type Q of Ito [2].

Moreover suppose that A_0, A_1, B_0 and B_1 are symmetric, then the second condition is trivial and the first condition is exactly the Williamson equation. If the Williamson equation is valid, then there exists an Hadamard matrix of the Williamson type.

2. The Paley type 1 matrix

Next we prove that the Paley type 1 matrix can be changed into the form of a C-matrix of GQ type. The Paley type 1 matrix is defined as follows (see [7]).

Definition. Let q be a prime power, $q \equiv 3 \pmod 4$, $F = \mathrm{GF}(q)$ the finite field of q elements, $K = \mathrm{GF}(q^2)$ a quadratic extension over F, and K^* and F^* the multiplicative groups of K and F respectively. Furthermore let η be a generator of K^*, $\gamma = \eta^{(q+1)/2}$, and let $N_{K/F}$ and $S_{K/F}$ denote the relative norm and relative trace from K to F respectively. Denote by ψ the quadratic character of F. Then the matrix

$$P = (\psi(N_{K/F}\alpha)\psi(S_{K/F}\gamma^{-1}\beta\alpha^{-1}))_{\alpha,\beta\in K^*/F^*}$$

is called the *Paley type* 1 *matrix*.

We recall here the definition of Seidel-equivalence of matrices.

Definition. If a square matrix A can be obtained from a square matrix B by a sequence of two kinds of operations:

(i) multiplying the row and the corresponding column by -1 simultaneously,
(ii) interchanging two rows and the corresponding two columns simultaneously,

then A will be said to be *Seidel-equivalent* to B.

Theorem 1. *The Paley type* 1 *matrix is Seidel-equivalent to a C-matrix of* GQ *type with some additional properties*:

(i) $\mathcal{A}$ *is skewsymmetric,*
(ii) $B_{2N-m-1} = -B_m^*$ *for* $m = 0, \ldots, N-1$ *where* $q+1 = 2^{s+1}n$, $s \geqslant 1$, n *odd,* $N = 2^{s-1}$.

Proof. Let $\mathcal{H} = \{\eta^{4Nk+2nk'}: k = 0, \ldots, n-1, k' = 0, \ldots, 2N-1\}$. Then we can take $\mathcal{H} \cup \mathcal{H}\eta^n$ as a complete system of representatives of K^*/F^*. When we put

$$w_k = \psi(S_{K/F}\gamma^{-1}\eta^k),$$

the matrix P can be written in the following form by using w_k:

$$P = \begin{pmatrix} (w_{4N(-k+l)+2n(k'+l')}) & (w_{4N(-k+l)+2n(k'+l')+n}) \\ (-w_{4N(-k+l)+2n(k'+l')-n}) & (-w_{4N(-k+l)+2n(k'+l')}) \end{pmatrix}_{\substack{k,l=0,\ldots,n-1 \\ k',l'=0,\ldots,2N-1}}.$$

We see that

$$\begin{cases} w_{-k} = w_k & \text{if } k \text{ is odd,} \\ w_{-k} = -w_k & \text{if } k \text{ is even,} \end{cases} \tag{2}$$

$$w_{k+4nN} = -w_k. \tag{3}$$

Furthermore if we write

$$A_m = (w_{4N(-k+l)+2mn})_{k,l=0,\ldots,n-1},$$
$$B_m = (w_{4N(-k+l)+(2m+1)n})_{k,l=0,\ldots,n-1},$$
$$C_m = (w_{4N(-k+l)+(2m-1)n})_{k,l=0,\ldots,n-1},$$

then these are circulant matrices of order n, and it follows from (2) that

$$A_{m+2N} = -A_m, \quad A_{2N-m} = A_m^*, \quad A_{-m} = -A_m^*, \quad B_{m+2N} = -A_m, \quad B_{2N-m} = -C_m^*.$$

Thus the matrix P take the form:

$$P = \begin{pmatrix} \mathscr{A} & \mathscr{B} \\ -\mathscr{B}^* & \mathscr{A}^* \end{pmatrix},$$

where $\mathscr{A} = (A_m)_{m=0,\ldots,2N-1}$, and $\mathscr{B} = (B_m)_{m=0,\ldots,2N-1}$. Notice that $A_{m+2N} = -A_m$, $B_{m+2N} = -B_m$, i.e. $\mathscr{A}$, $\mathscr{B}$ are negacyclic. And since $A_{2N-m} = A_m^*$, it is obvious that $\mathscr{A}$ is skewsymmetric.

On the other hand, from (2), (3) we have

$$\begin{aligned} B_{2N-m-1} &= (w_{4N(-k+l)+(2(2N-m-1)+1)n}) = (w_{4N(-k+l)+4nN-(2m+1)n}) \\ &= (-w_{4N(-k+l)-(2m+1)n}) = (-w_{-4N(-k+l)+(2m+1)n}) = -B_m^*, \end{aligned}$$

which completes the proof of Theorem 1. □

3. Infinite series of Hadamard matrices of generalized quaternion type

In this section, we construct some infinite series of Hadamard matrices of GQ type.

Let q be a power of a prime p, $F = \mathrm{GF}(q)$ denote a finite field of q elements, $K = \mathrm{GF}(q^t)$ an extension of F of degree t, $t \geq 2$. Let η be a generator of K^* and let S_K and S_F denote the absolute trace in K and F. Furthermore let $S_{K/F}$ and $N_{K/F}$ be the relative trace and relative norm from K to F respectively.

Definition. Let χ be a character of F and $\zeta_p = \mathrm{e}^{2\pi i/p}$, then the Gauss sum $\tau_F(\chi)$ is defined by

$$\tau_F(\chi) = \sum_{\alpha \in F} \chi(\alpha) \zeta_p^{S_F \alpha}.$$

If χ is a nonprincipal character of K, then the ratio

$$\theta_\chi = \frac{\tau_K(\chi)}{\tau_F(\chi)}$$

of two Gauss sums is called the *relative Gauss sum associated with* χ.

The following theorem on the relative Gauss sum is very important and it is a key point of this paper.

Theorem 2. *Suppose that χ is a character of K inducing in F a nonprincipal character. Then the relative Gauss sum associated with χ can be written in the following form*

$$\theta_\chi = \sum_{\alpha \in K^*/F^*} \chi(\alpha)\bar{\chi}(S_{K/F}\alpha),$$

and we have the norm relation

$$\theta_\chi \bar{\theta}_\chi = q^{t-1}.$$

Proof. See [8]. □

Using Theorem 2 for the case $t=2$, we give infinite series of Hadamard matrices of GQ type.

Theorem 3. *Let $q+1 = 2^s n$, $s \geq 2$, n odd, ρ a primitive 2^{s+1}th root of unity and ω an arbitrary nth root of unity. Put $\chi = \chi_{2^{s+1}}\chi_n$ where $\chi_{2^{s+1}}(\eta) = \rho$, $\chi_n(\eta) = \omega$, so that χ induces a quadratic character ψ in F.*

Then for the relative Gauss sum θ_χ we have

$$\theta_\chi = \alpha + \beta\rho^n, \quad \alpha, \beta \in \mathbb{Z}[\rho^2, \omega],$$

and the right regular representation matrix of

$$\gamma = \alpha \pm i + \beta j$$

gives an Hadamard matrix of GQ *type of order $2^s n$ where i is a primitive fourth root of unity.*

Proof. By Theorem 2, we have

$$\theta_\chi = \sum_{m=0}^{q} \chi(\eta^m)\psi(S_{K/F}\eta^m) = \sum_{m=0}^{q} \psi(S_{K/F}\eta^m)\rho^m\omega^m.$$

Dividing in two partial sums according as m even or m odd, we get

$$\theta_\chi = \sum_{m=0}^{(q-1)/2} \psi(S_{K/F}\eta^{2m})\rho^{2m}\omega^{2m} + \left(\sum_{m=0}^{(q-1)/2} \psi(S_{K/F}\eta^{2m+n})\rho^{2m}\omega^{2m}\right)\rho^n = \alpha + \beta\rho^n,$$

where

$$\alpha = \sum_{m=0}^{(q-1)/2} \psi(S_{K/F}\eta^{2m})\rho^{2m}\omega^{2m} \quad \text{and} \quad \beta = \sum_{m=0}^{(q-1)/2} \psi(S_{K/F}\eta^{2m+n})\rho^{2m}\omega^{2m}.$$

Similarly, we get

$$\bar{\theta}_\chi = \sum_{m=0}^{q} \psi(S_{K/F}\eta^m)\rho^{-m}\omega^{-m} = \sum_{m=0}^{q} (-1)^m\psi(S_{K/F}\eta^m)\rho^m\omega^m = \alpha - \beta\rho^n,$$

so that $\bar{\alpha} = \alpha$, $\overline{\beta\rho^n} = -\beta\rho^n$. We obtain finally

$$\begin{aligned}\theta_\chi\bar{\theta}_\chi &= (\alpha + \beta\rho^n)(\bar{\alpha} + \overline{\beta\rho^n}) = \alpha\bar{\alpha} + \beta\rho^n\overline{\beta\rho^n} + \alpha\overline{\beta\rho^n} + \beta\bar{\alpha}\rho^n \\ &= \alpha\bar{\alpha} + \beta\bar{\beta} + \alpha(-\beta\rho^n) + \alpha\beta\rho^n = \alpha\bar{\alpha} + \beta\bar{\beta} = q.\end{aligned}$$

Since coefficients of γ on the basis $\mathscr{L}$ are from $\{1, -1\}$, and $\mathcal{N}(\alpha \pm i + \beta j) = q + 1$, the right regular representation matrix of γ is an Hadamard matrix of GQ type of order $2^s n$. □

In the special case $s = 2$ we obtain an infinite series of Hadamard matrices of GQ type of order $4n$. More generally, if there exists an element $X = \alpha + \beta\rho^n$ of $\mathbb{Z}[\rho, \omega]$, satisfying

(i) all coefficients on the basis $\mathscr{L}$ are from $\{1, -1\}$, and

(ii) $\mathcal{N}(X) = 2^s n$,

then by putting

$$\gamma = \alpha + \beta j,$$

we get an Hadamard matrix of GQ type of order $2^s n$. The problem to find X with the above properties in the cyclotomic field $Q(\rho, \omega)$ has been studied by many number theorists, but it seems very difficult in general.

Corollary to Theorem 3. *Let α, β be as in Theorem 3. Then the right regular representation matrix of*

$$\gamma = (\alpha - i + \beta\rho^n j)(1 - j) = (1 - j)(\theta_\chi + ij)$$

is an Hadamard matrix of GQ *type of order $2^{s+1}n$. In particular, if $s = 1$, then we get an Hadamard matrix of Turyn's type* [4].

Proof. It is obvious that the coefficients of γ on the basis $\mathscr{L}$ are from $\{1, -1\}$ and $\mathcal{N}(\gamma) = 2^{s+1}n$. For $s = 1$, we have $q + 1 = 2n$, n odd, and

$$\theta_\chi = \sum_{m=0}^{q} \psi(S_{K/F}\eta^m) i^m \omega^m = \sum_{m=0}^{n-1} (\psi(S_{K/F}\eta^m) i^m + \psi(S_{K/F}\eta^{m+n}) i^{m+n}) \omega^m.$$

Since n is odd, we have

$$\theta_\chi = \psi(2) + \sum_{m=1}^{n-1} (\psi(S_{K/F}\eta^{4m}) i^m + i^n \psi(S_{K/F}\eta^{4m+n})) \omega^m.$$

Write $a_m = \psi(S_{K/F}\eta^{4m}) + i^n \psi(S_{K/F}\eta^{4m+n})$ for $m \geqslant 1$, then it is easy to verify that

$$a_{-m} = a_m \quad \text{for } m = 1, \ldots, (n-1)/2.$$

In fact

$$\begin{aligned}
\psi(S_{K/F}\eta^{-4m}) &= \psi(S_{K/F}\eta^{4m}) / \psi(N_{K/F}\eta^{4m}) = \psi(S_{K/F}\eta^{4m}). \\
\psi(S_{K/F}\eta^{-4m+n}) &= \psi(S_{K/F}\eta^{4m-n}) / \psi(N_{K/F}\eta^{4m-n}) \\
&= \psi(S_{K/F}\eta^{4m+n}) \psi(\eta^{-2n}) / \psi(N_{K/F}\eta^{-n}) \\
&= \psi(S_{K/F}\eta^{4m+n})(-1)^{-1}/(-1)^{-n} = \psi(S_{K/F}\eta^{4m+n}).
\end{aligned}$$

This leads to an Hadamard matrix of Turyn's type (more precisely see [8]). □

Theorem 4. *Let $q+1=2n$, n odd and ρ a primitive octic root of unity. Let η and ω be as in Theorem 3. Put $\chi=\chi_8\chi_n$, $\chi_8(\eta)=\rho$. So that χ induces a biquadratic character in F.*

The right regular representation matrix of

$$\gamma=(\theta_\chi+\rho^t j)(1+i)(1+j),\quad t=1,3,5,7,$$

gives an Hadamard matrix of GQ *type of order $8n$. We may change the order of factors $\theta_\chi+\rho^t j$, $1+i$ and $1+j$ arbitrarily.*

Proof. We have

$$\theta_\chi=\sum_{m=0}^{q}\bar{\chi}(S_{K/F}\eta^m)\rho^m\omega^m=\sum_{m=0}^{n-1}\{\bar{\chi}(S_{K/F}\eta^{8m})+\rho\bar{\chi}(S_{K/F}\eta^{8m+n})\}\omega^m,$$

and

$$\theta_\chi\bar{\theta}_\chi=q.$$

Put

$$\gamma=(\theta_\chi+\rho j)(1+i)(1+j)=\theta_\chi(1+i)-\rho+\rho^3+(\theta_\chi(1+i)+\rho-\rho^3)j.$$

We proceed similarly in the cases where the order of three factors is changed and in the cases where $t=3$, 5, or 7.

Now the coefficient of ω^m with $m=0$ in $\theta_\chi(1+i)$ is $\bar{\chi}(S_{K/F}1)(1+i)=\pm1\pm\rho^2$, and the coefficients of γ on the basis $\{\rho^k\omega^l,\ \rho^k\omega^l j;\ 0\leqslant k\leqslant 3,\ 0\leqslant l\leqslant n-1\}$ are all from $\{1,-1\}$, and γ satisfies

$$\mathcal{N}(\gamma)=\mathcal{N}(\theta_\chi+\rho j)\mathcal{N}(1+i)\mathcal{N}(1+j)=(q+1)\cdot 2\cdot 2=4(q+1).\quad\square$$

On the other hand, if there exists an Hadamard matrix of GQ type of order $2^s n$, we can double its order.

Theorem 5. *Assume that the right regular representation matrix of $\xi=\alpha+\beta j$ in $\mathcal{R}$ is an Hadamard matrix of* GQ *type of order $2^s n$. Let ρ be a primitive 2^{s+1}th root of unity. Then*

$$\gamma=(\alpha+\beta j)(1+\rho^t j)\quad \textit{for}\quad t=1,3,5,\ldots,2^s-1,$$

generates an Hadamard matrix of GQ *type of order $2^{s+1}n$. We can exchange the order of two factors $\alpha+\beta j$ and $1+\rho^t j$.*

Proof. Consider

$$\gamma=(\alpha+\beta j)(1+\rho j)=\alpha-\beta\rho+(\beta+\alpha\rho)j.$$

We proceed similarly in the cases where $t=3,5,\ldots,2^s-1$ and in the case where $\gamma=(1+\rho^t j)(\alpha+\beta j)$.

Now since ξ generates an Hadamard matrix of GQ type, we have that the coefficients of $\alpha-\beta\rho$ or $\beta+\alpha\rho$ on the basis $\{\rho^k\omega^l\colon 0\leqslant k\leqslant 2^s-1,\ 0\leqslant l\leqslant n-1\}$

are from $\{1, -1\}$, and γ satisfies

$$\mathcal{N}(\gamma) = \mathcal{N}(\alpha + \beta j)\mathcal{N}(1 + \rho j) = 2^s n \cdot 2 = 2^{s+1} n. \quad \square$$

Hence starting from an Hadamard matrix of GQ type of order $4n$ or $8n$ constructed in Theorems 3 and 4, we can have an infinite series, on 2^s, of Hadamard matrices of GQ type.

4. Hadamard matrices of generalized quaternion type of order 24

We have calculated all the Hadamard matrices of GQ type of order 24 based on the following equivalence relation. If an element ξ in $\mathcal{R}$ can be mapped to an element η in $\mathcal{R}$ by successive application of the following operations:

(i) $\xi \to \xi^\sigma$, $\sigma \in \text{Aut } G$, and

(ii) $\xi \to \alpha\xi$, $\alpha \in G$,

then the Hadamard matrix generated by ξ is said to be equivalent to the Hadamard matrix generated by η.

Neglecting the possibility of transposes we have a total of eight classes. We were able to give the construction for seven of the classes but the remaining one class has proved elusive. Two classes, 2.3, 3.1, are generated by Theorems 4 and 3 respectively, and all other Hadamard matrices of order 24 except class 3.2 are constructed from Hadamard matrices of order 12 by using the following theorem.

Theorem 6. *Let ρ be a primitive octic root of unity, ζ_n an arbitrary nth root of unity and $\sigma_3 : \rho \to \rho^3$, $\sigma_5 : \rho \to \rho^5 = -\rho$, $\sigma_7 : \rho \to \rho^7 = -\rho^3$ automorphisms of $Q(\rho)$. For*

$$\alpha = a_0 + a_1\rho + a_2\rho^2 + a_3\rho^3, \quad a_i \in \mathbb{Z}[\zeta_n],$$

let

$$f(\alpha) = \bar{a}_0 a_2 + \bar{a}_1 a_3 - \bar{a}_2 a_0 - \bar{a}_3 a_1, \qquad g(\alpha) = \bar{a}_0 a_1 + \bar{a}_1 a_2 + \bar{a}_2 a_3 - \bar{a}_3 a_0$$

then we have

$$\alpha\bar{\alpha} = a_0\bar{a}_0 + a_1\bar{a}_1 + a_2\bar{a}_2 + a_3\bar{a}_3 + f(\alpha)\rho^2 + g(\alpha)\rho + \overline{g(\alpha)\rho}.$$

Suppose that one of the following conditions is satisfied:

(i) $f(\alpha) = 0$, *i.e.* $a_0 + a_2 i + a_1 j + a_3 ij$ *generates an Hadamard matrix of* GQ *type of order* $4n$,

(ii) $f(\alpha) = 0$ *and* $g(\alpha) = \overline{g(\alpha)}$, *and*

(iii) $g(\alpha) = 0$.

In each case we have an Hadamard matrix of GQ *type of order* $8n$ *generated by*

(i) $\alpha + \alpha^{\sigma_5} j$, $\alpha + \bar{\alpha}^{\sigma_5} j$,

(ii) $\alpha + \alpha^{\sigma_3} j$, $\alpha + \bar{\alpha}^{\sigma_3} j$,

(iii) $\alpha + \alpha^{\sigma_3} j$, $\alpha + \alpha^{\sigma_7} j$, $\alpha + \bar{\alpha}^{\sigma_3} j$, $\alpha + \bar{\alpha}^{\sigma_7} j$, *respectively.*

Notation. By $(\alpha_0, \alpha_1, \alpha_2, \alpha_3)(\beta_0, \beta_1, \beta_2, \beta_3)$ we denote eigenvalues of the component matrices $A_0, A_1, A_2, A_3, B_0, B_1, B_2, B_3$ respectively. We denote a cubic root of unity by ω.

Table 1. Classes of Hadamard matrices of GQ type of order 24 and their method of construction

1. When the eigenvalues are $(3,1,1,1)(3,-1,1,-1)$ for $\omega=1$, then for $\omega\neq 1$ they are:		
Class 1.1	$(0,-2,-2,-2)(0,2,-2,2)$	By applying σ_5, obtained from an Hadamard matrix of the Williamson type of order 12.
Class 1.2	$(0,-2,-2\omega,-2)(0,2,-2\omega,2)$	By applying σ_5, obtained from an Hadamard matrix of GQ type of order 12 constructed by Theorem 3.
Class 1.3	$(0,-2,-2\omega,-2)(0,2,-2\omega^2,2)$	By applying $\tilde{\sigma}_5$, obtained from an Hadamard matrix of GQ type of order 12 constructed by Theorem 3.
2. When the eigenvalues are $(3,1,1,-1)(3,-1,1,1)$ for $\omega=1$, then for $\omega\neq 1$ they are:		
Class 2.1	$(0,-2,-2,2)(0,2,-2,-2)$	By applying σ_5, obtained from an Hadamard matrix of the Williamson type of order 12.
Class 2.2	$(0,-2,-2\omega,2)(0,2,-2\omega,-2)$	By applying σ_5, obtained from an Hadamard matrix of GQ type of order 12 constructed by Theorem 3.
Class 2.3	$(0,-2,-2\omega^2,2\omega)(0,2\omega,-2\omega^2,-2)$	Obtained by Theorem 4 for the case $q=5$.
3. When the eigenvalues are $(3,3,-1,1)(1,1,-1,1)$ for $\omega=1$, then for $\omega\neq 1$ they are:		
Class 3.1	$(0,0,2\omega,-2)(-2\omega^2,-2,2,-2\omega)$	Obtained by Theorem 3 for the case $q=23$.
Class 3.2	$(0,0,2,-2)(-2\omega,-2\omega^2,2,-2)$	The method of construction is not known to the author.

References

[1] P. Delsarte, J.M. Goethals and J.J. Seidel, Orthogonal matrices with zero diagonal II, Canad. J. Math. 23 (1971) 816–832.

[2] N. Ito, Note on Hadamard matrices of type Q, Studia Scientiarum Mathematicarum Hungarica 16 (1981) 389–393.

[3] S. Lang, Cyclotomic Fields (Springer, Berlin, 1978).

[4] R.J. Turyn, An infinite class of Williamson matrices, J. Combin. Theory Ser. A 12 (1972) 319–321.

[5] W.D. Wallis, A.P. Street and J.S. Wallis, Combinatorics: Room squares, sum-free sets, Hadamard matrices, Lecture Notes in Math. 292 (Springer, Berlin, 1972).

[6] J. Williamson, Hadamard's determinant theorem and sum of four squares, Duke Math. J. 11 (1944) 65–81.

[7] K. Yamamoto, On a generalized Williamson equation, Colloquia Mathematica Societatis Janos Bolyai 37 (1985) 839–850.

[8] K. Yamamoto and M. Yamamda, Williamson Hadamard matrices and Gauss sums, J. Math. Soc. Japan 37 (1985) 703–717.

[9] N. Ito, J.S. Leon and J.Q. Longyear, Classification of 3-(24, 12, 15) designs and 24-dimensional Hadamard matrices, J. Combin. Theory Ser. A 31 (1981) 66–93.